REPRESENTATION OF FOURIER SERIES

Fourier series- The representation of signal over a certain interval of time in terms of the linear combination of orthogonal function is called Fourier series. Fourier ~~series~~ series is applicable only for periodic signals.

There are three types (form). of Fourier series

i) Trigonometric form.

ii) Cosine form.

iii) Exponential form.

If the orthogonal function are trigonometric function then it is called trigonometric Fourier series.

If the orthogonal function are exponential function then it is called exponential Fourier series.

Existence of Fourier series:

The condition under which a periodic signal can be represented by a Fourier series are known as Dirichlet condition.

In each period

1) The function $x(t)$ must be a single valued function

2) The function $x(t)$ has only a finite number of maxima and ~~minima~~ minima.

3) The function $x(t)$ has a finite number of discontinuities.

4) The function $x(t)$ is absolutely integrable over one period i.e

$$\int_{0}^{T} |x(t)|\, dt < \infty$$

The condition 4 is also known as the weak Dirichlet condition.

The conditions 1 and 3 are known as strong Dirichlet conditions.

* If the function (periodic signal), satisfies the weak Dirichlet condition, the existence of Fourier series is guaranteed. But the series may not converge at every point. For strong condition satisfied, the convergence is also guaranteed.

TRIGONOMETRIC FORM OF FOURIER SERIES

Let us consider a periodic signal,

$$x(t) = A \sin \omega_0 t \quad \text{with period } T = \frac{2\pi}{\omega_0}$$

Here we can show that a signal $x(t)$, a sum of sine and cosine functions whose frequencies are integral multiples of ω_0 is a periodic signal.

Let the signal $x(t)$ is expressed as,

$$x(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + \dots a_k \cos k\omega_0 t + b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + \dots + b_k \sin k\omega_0 t.$$

i.e.

$$x(t) = a_0 + \sum_{n=1}^{k} \cancel{a_n \cos\omega_0 t}\; a_n \cos\omega_0 n t + b_n \sin n\omega_0 t \quad \text{--- (1)}$$

where
$a_0, a_1, a_2 \dots a_k$ and
$b_0, b_1, b_2 \dots b_k$ are constants

ω_0 = Fundamental frequency

As we know that ~~x(t)~~ if a signal is periodic, then it must satisfy the following condition

$$x(t) = x(t+T) \quad \text{for all } T$$

as

$$x(t) = x(t+T) = a_0 + \sum_{n=1}^{k} a_n \cos\omega_0 n(t+T) + b_n \sin\omega_0 n(t+T)$$

$$= a_0 + \sum_{n=1}^{k} a_n \cos\omega_0 n\left(t + \frac{2\pi}{\omega_0}\right) + b_n \sin\omega_0 n\left(t + \frac{2\pi}{\omega_0}\right)$$

$$\left[\because T = \frac{2\pi}{\omega_0}\right]$$

Evaluation of Fourier coefficients of the Trigonometric Fourier Series

Calculation of a_0 To evaluate a_0, integrate $x(t)$ ~~over~~ on the both sides of $x(t)$ over one period t_0 to t_0+T.

$$\therefore \int_{t_0}^{t_0+T} x(t) = \int_{t_0}^{t_0+T} a_0 + \int_{t_0}^{t_0+T} \left[\sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]\right]$$

$$= a_0[\cancel{t_0} + T - \cancel{t_0}] + \int_{t_0}^{t_0+T} \left[\sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right]$$

$$= a_0 T + \sum a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t + b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t$$

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 T + 0 + 0$$

$$\because a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t = 0$$

$\because$ the net area of sinusoidal wave over complete period is 0.

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\cdot dt$$

Thus,

$$\boxed{a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\cdot dt}$$

Calculation of a_n

$\because$ we know that

$$x(t) = a_0 + \int_{t_0}^{t_0+T} \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

On multiplying on the both side of ~~above equation ① we get~~ the equation ① by $\cos m\omega_0 t$ we get:

$$\int_{t_0}^{t_0+T} x(t)\cdot \cos m\omega_0 t\,dt = a_0 \int_{t_0}^{t_0+T} \cos m\omega_0 t\,dt + \sum a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t \cdot \cos m\omega_0 t + \int_{t_0}^{t_0+T} \sin n\omega_0 t \cdot \cos m\omega_0 t\,dt$$

$\therefore$ The value of $\displaystyle\int_{t_0}^{t_0+T} \cos n\omega_0 t \, \cos m\omega_0 t \, dt = \begin{cases} 0 & \text{if for } m \neq n \\ T/2 & \text{for } m = n \neq 0. \end{cases}$

$$\int_{t_0}^{t_0+T} \sin n\omega_0 t \cdot \sin m\omega_0 t \, dt = \begin{cases} 0 \; ; & \text{for } m \neq n \\ T/2 \; ; & \text{for } m = n \neq 0. \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin n\omega_0 t \cdot \cos m\omega_0 t \, dt = 0 \quad \text{for all values of } m \text{ and } n.$$

On substituting the value of above values in equation (1) we get-

$$\int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t \, dt = \underbrace{\int_{t_0}^{t_0+T} a_0 \cos m\omega_0 t \, dt}_{0} + \sum_{n=1}^{\infty} \int_{t_0}^{t_0+T} a_n \cos n\omega_0 t \cdot \cos m\omega_0 t + \underbrace{\int_{t_0}^{t_0+T} \sum_{n=1}^{\infty} \sin n\omega_0 t \cos m\omega_0 t}_{0}$$

$$= 0 + \sum_{n=1}^{\infty} \int_{t_0}^{t_0+T} (a_n \cos n\omega_0 t \cdot \cos m\omega_0 t) \, dt$$

$$\int_{t_0}^{t_0+T} x(t) \cdot \cos m\omega_0 t = \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} (\cos n\omega_0 t \cdot \cos m\omega_0 t) \, dt$$

At $m = n$, $= a_m \cdot T/2$

$$\therefore a_m = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos m\omega_0 t$$

$$\boxed{\therefore a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega_0 t}$$

Calculation of b_n

Multiply equation ① by $\cancel{\quad}$ $\sin m\omega_0 t$ on the both sides we get,

$$\therefore \int_{t_0}^{t_0+T} x(t)\cdot \sin m\omega_0 t\, dt = \int_{t_0}^{t_0+T} a_0 \sin m\omega_0 t + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t \cdot \sin m\omega_0 t\, dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t \cdot \sin m\omega_0 t\, dt$$

On substituting the value of first and (II) second term equal to 0 because, the area under these integrals is equal to 0.

$$\therefore \int_{t_0}^{t_0+T} x(t)\cdot \sin m\omega_0 t\, dt = 0 + 0 + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t \cdot \sin m\omega_0 t\, dt$$

$$\int_{t_0}^{t_0+T} x(t)\cdot \sin m\omega_0 t\, dt = b_m T/2. \quad [\text{at } m = n]$$

$$\therefore b_m = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cdot \sin n\omega_0 t\, dt \quad [\because m = n]$$

Then

$$\boxed{b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cdot \sin n\omega_0 t\, dt}$$